Arithmetic Dynamics of Polynomials in $\mathbb{Z}[x]$

Summer Project Report - Julia Stadlmann

Take a polynomial $f(x)$ with integer coefficients and an integer a_0. Consider the sequence $a_{n+1} = f(a_n)$. What properties do the elements of this sequence have? How many primes appear in it? For linear $f(x)$ this question was settled long ago - Dirichlet's theorem states that the iterative sequence (a_n) contains infinitely many primes, provided a_0 and $f(0)$ are coprime. For higher degree polynomials, however, the properties of (a_n) are somewhat of a mystery. This summer I had the opportunity to do a project on arithmetic dynamics under the supervision of Prof. Erban, studying the different approaches to this problem.

I soon came across a related structure - sequences of Mahler measures of iterated polynomials. (The Mahler measure of a polynomial is the modulus of the product of all roots lying outside the unit circle.) Such sequences are notable for the large primes appearing in them, but my attention was drawn to a frequently mentioned, somewhat curious conjecture: Lehmer's conjecture proposes that the Mahler measure of monic non-cyclotomic polynomials is bounded below by a value greater 1. My interest was piqued because, simply put, intuitively it felt like this should just be wrong. I wanted to learn more about this problem, so much so that Mahler measures became the main focus of my project after the first few weeks.

Initially, I conducted some simple computational searches with the goal of generating numerical data and maybe, quite optimistically, finding a counterexample. Simultaneously, I read a number of papers on the subject - as it turns out, the conjecture has been proved for a number of different classes of polynomials, for example those with all coefficients odd, characteristic polynomials of symmetric integer entry matrices and, maybe most importantly, those whose coefficients do not read the same from back to front. Through examining these proofs and my own investigations, I slowly started to gain a much better understanding of the problem at hand, the distribution of roots of polynomials, and gradually changed my mind and became reasonably convinced of the conjecture. Overall, this was a very exciting process to go through.

Additionally, my summer project also opened my eyes to how varied the approaches to a single question can be. Many papers I read had an algebraic flavour, but others were entirely combinatorial or used geometry, and some even involved differential equations. It was interesting to study all these different methods, to develop my own ideas based on them and to try proving something myself. There were many ups and downs, and only few of my proofs worked in the end, but each failure helped further my understanding of the subject and was valuable in some way.

The project also helped confirm that mathematical research really is what I want to do in future. Through it I became much more exposed to the many different approaches to number theory, and it helped broaden my horizon for what directions I could take for my PhD. It was a truly wonderful experience and I would to thank Merton College for its support.