Ehud Hrushovski

Ton Yeh

Ehud Hrushovksi is a Professor of Mathematical Logic and Fellow of Merton College, regarded as a leading model theorist of the 21st century. What follows is the interview that I had with him in July 2018, with my questions in boldface.

What recollections have you of mathematics at school?

A few. I had no idea that one could be a mathematician. I had no thoughts like that. Looking back, I was very struck by certain mathematical things. I remember very well learning the proof of commutativity of multiplication, being shown a rectangle divided into small squares. I remember even being skeptical about it, being unsure whether one was counting the corners of the squares, or the actual squares, whether there was a cheat there. And I remember the real line being introduced, and the revelation of some connection between numbers and something drawn. On the whole, though, it was pretty dreary and boring. It was all about learning things by heart. But from time to time, I was struck by certain ideas. Much later in high school I had a very good teacher who was in fact doing a project on maths education.

When did you decide to pursue mathematics?

I first wanted to learn mathematics and philosophy, and I was very excited about that. I had in mind, going into a much more humanities direction, that perhaps it would be good to learn something formal. And it was in Year 1 of Maths and Philosophy that I fell in love with maths. As for philosophy, at the time, though I was very interested in the questions, the experience was less good. I thought I should go deeper in the mathematics direction. Without ever losing interest in the philosophical questions.

Would you consider your work to have any philosophical implications?

It's a question of scale. Sometimes one is doing fairly technical things, and sometimes one feels that there is a contribution to something bigger. I don't believe in the demarcation of domains. That's really something for administration, not something intrinsic. The questions are those which I view as foundational. Nevertheless, at any given time or on any given day, in order to pursue them, the way forward is to go through what would seem to an outsider to be extremely technical details. That's true at any scale. When one works within mathematics, it doesn't matter if it's different fields or one field, one may still have a big goal in mind, but at a given time, one may be working on a small lemma.

When did you first come into contact with model theory?

It was through a friend – I was in Israel at the time – a few years after I did Maths and Philosophy here (at Brasenose in 1976/7). I had heard vaguely about Tarski, about the formalisation of languages, from philosophy. It was a really mysterious world. Set theory was much easier and made sense very well. Model theory does not. Set theory is much easier for philosophically-oriented people. It connects well with a certain number of ideas that are not far from where they come from in the first place, because you have certainty, validity of argument, etc. Whereas model theory has equally many philosophical issues, but different and strange.

Mathematics involves a lot of things that you wouldn't have guessed. And model theory involves a lot of that. It means understanding the structure of actual mathematics, as opposed to some theoretical boundary across maths. Actual maths is arguably cut out by boundaries depending on different ideas, not just on ideas underlying Gdel's theorem, abstraction and so on. It's more foreign, it's harder to get into, but I think it's not a bad idea for philosophers of mathematics to know a bit more of that.

When you entered model theory, would you say that it was still in its early stages?

No. I finished my PhD in 1986 so I suppose I started working on it in 1982. Model theory was rather mature at that point. It's true that it was a young subject. Significant things could be seen or guessed by the 1950s certainly, there was already Tarski's theorem, the compactness theorem, LwenheimSkolem theorem. Most of what we manage to teach today about model theory, at Oxford, was known by the 50s. By the time I was a student, further things had happened. Model theory by its nature is theory 'of' something. Robinson called it 'metamathematics of algebra', or one could say of some other theories, but it's always of something, it needs to look at large mathematical fields but nevertheless fields that are not pure mathematics in abstraction. To do that requires a lot of work. You cannot just come and look, you have to adapt it to logical methods. To do this, you have to do it one field at a time. Tarski did it for Greek geometry. Then it needed to be done for other areas, and by the time I came along it had been done for many things, for finite fields, for differential fields, each one of these captures a field of mathematics. That I was much less aware of back then. What I was aware of was a huge revolution by Saharon Shelah, who turned it from a nice collection of ideas to a really deep area with its own systematic methodology at a completely different order of magnitude. This is stability theory. All that had been achieved before I came along.

At school, one is told that the foundations of mathematics are certain axioms in set theory, and everything else just follows. How would you modify that picture for a first-year undergraduate?

The existence of such a foundation is a fantastic achievement, which gives a common language and common standard of precision to mathematics. It should however not be taken out of context. It's not mathematics itself. In many ways it is itself connected to interesting pieces of mathematics. The fact that you can embed an existing mathematical theory in that, is a plus. The theory existed before that. Vast amounts of mathematical ideas existed long before the formalisation in the 19th century. Ideas, like those around 17th-century analysis, can be embedded in set theory, but that doesn't mean that they can be identified with the formal letters that purport to represent them. Certain parts of them are also computational and once again, the computational aspect does not at all exhaust the interest in the ideas. If your initial interests are just that, then you would not have discovered any of the main ideas in 20th-century mathematics.

What would be an example of model theory that can be explained to a beginner?

The easiest concrete example of model theory to explain is Tarski's study of Greek geometry, which is a rich subject. It appears to be about certain figures in the plane, cut out by some algebraic equations. It's a great achievement to do one theorem at a time, to prove the Pythagorean theorem or the great many theorems proven by the Greeks, each one a separate achievement. At a certain point one gets the feeling that there's also a domain, something closed there, where you could describe the whole family of theorems and maybe systematise them. So we have Tarski's theorem about the decidability of elementary geometry. Elementary geometry means that any question we can ask about algebraic shapes, including conic sections, hyperbolas, parabolas, or also more complicated ones, in fixed dimension and fixed degree, can be answered systematically. Descartes had a feeling that it would work, that he could somehow do analytic geometry and answer questions like that, with algebraic manipulations. Tarski proved that you can systematically answer all these things. That sheds a different light on the whole field; it's one whole level of abstraction up from a single elementary theorem in geometry. And surprisingly it reflects back once you have a global understanding of the area, that you can also tackle concrete questions using that – there were specific questions to which people were looking for answers – about complicated algebraic shapes, singularities and so on. Having the systematic understanding which Tarski had is the beginning of model theory. It gives, e.g., ways to prove effectiveness results but also to prove numerical bounds of the questions. Sometimes a specific question is so difficult that you can only get a handle on them via a more abstract and wider view.

What are the big questions in model theory today?

It's difficult to give a one-sentence answer to this! One needs to explain the development of the subject. So let me just say in a very, very general way, there are many frontiers. As I said, the idea of trying to capture all of mathematics in one logical system has limitations, and is not really possible, by Gdel's theorem. Model theory tries to capture large but not universal parts of mathematics. That entails doing it one at a time. So you have to work very closely, you look for other complete systems, 'tame' systems as we call them, to capture other large parts of mathematics. With respect to some of them, there is a lot of understanding. With respect to some, there's no understanding at all. There are beautiful areas of number theory with theoretical undecidability, but practical decidability in that for hundreds of years, people attack questions which get harder and harder, and they succeed, they throw more and more light on them. It's not a random fact, it requires explanation. This explanation would be something similar to Tarski's theorem, but capturing large parts of number theory. There must be some parts of number theory which are axiomatisable, which are not Gdelian. One of the challenges is to do just that, to find other areas like that. In the other direction, we want to deepen the internal intrinsic tools of model theory. Once you've found such an area, what are you going to do in order to analyse it? You don't go proving theorems one by one on the basis of syntax. That you could have done before analysing it. You try instead to get a feeling for the nature of objects that live within the world. Of course, all mathematics does that, but model theory tries to do find its own tools which are different.