# Integer-valued polynomials and bounded functions on $p$-adic character varieties
# - Summer project report -

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In the summer of 2021, I carried out an 11 week project as part of a team of students lead by professor Konstantin Ardakov, from the Mathematical Institute. I am grateful to Merton College for the Summer project grant they awarded me for the first 8 weeks of this project. This allowed me to have one of the best experiences in my life and discover some wonderful Mathematics. I would also like to mention that I was helped financially by the Mathematics Institute, and I would like to express my gratitude towards them, too.

The main question of the project was whether the $o_L$-linear span of a certain set of polynomials $\sigma_{i,j}(Y)$ is the entirety of $\mathrm{Int}(o_L, o_L)$. Here, $L/\mathbb{Q}_p$ is a finite $p$-adic extension, $o_L$ is its ring of integers and $\mathrm{Int}(o_L, o_L)$ is the $o_L$-submodule of polynomials $f \in L[Y]$ that map any $x \in o_L$ to another element of $o_L$. The definition of the polynomials $\sigma_{i,j}(Y)$ is a bit technical, but is explicit, in the sense that one can compute each $\sigma_{i,j}(Y)$ (although there is no clear pattern in the coefficients). This question can be understood by an undergraduate that has been exposed to some $p$-adic numbers. However, it turns out to be surprisingly important. Prof. Ardakov has recently reduced an important question in rigid analytic geometry to two conditions, one of which is the problem stated above.

The first goal of the project was to understand the proof of the equivalence above. As I had just finished my second year, this was quite challenging. I spent more than half of the time reading a lot of lecture notes, chapters of books and previous research articles. However, this was extremely interesting and useful. I explored various algebraic subjects, from category theory and algebraic geometry, to rigid analytic geometry and Lubin-Tate theory. During this period I learned a lot and I got an idea not only of what was going on in the initial proof, but, maybe most importantly, what areas I find interesting and I would like to work on in the future.

After this initial reading phase, which took around 6-7 weeks, I was finally ready to understand the proof and attempt the actual problem. The structure of the $o_L$-module $\mathrm{Int}(o_L, o_L)$ is quite well understood: a set of polynomials $\{f_n : \deg(f_n) = n, n \geq 0\}$ is a basis if and only if the $\pi$-valuation of the leading term of each $f_n$ is $-w_q(n)$. This is called a Mahler basis. Here, $q$ is the residual degree of $L/\mathbb{Q}_p$ and $w_q(n) = \sum_{k\geq 1}\lfloor \frac{n}{q^k} \rfloor$. So, we needed to find such $f_n$'s in our module $M$, the $o_L$-linear span of the $\sigma_{i,j}$'s. A previous result shows that such an $f_n$ is in $M$ when $n$ is a power of $q$. We were able to extend this condition to $n$ having sum of digits in base $q$ at most $p-1$, using number theoretic techniques.

There were other approaches that we tried. The most promising one involved looking at the submodules $M_j$ of $M$, the $o_L$-linear spans of $\{\sigma_{i,j} : 0 \leq i \leq j\}$ for a fixed $j$. This greatly restricts the possible linear combinations, but in some sense it is easier to take linear combinations of $\sigma_{i,j}$'s with the same $j$. However, this proved to be unsuccessful, as computer-generated data suggested that $\bigcup_{j\geq 0} M_j \neq \mathrm{Int}(o_L, o_L)$. However, we worked on an algorithm that constructs regular bases for some other submodules of $M$, defined by $M^{(n)} := \{f \in M : \deg(f) \leq n\}$, for which $M = \bigcup_{n\geq 0} M^{(n)}$ by definition. The results suggest that, indeed, $M = \mathrm{Int}(o_L, o_L)$. Surprisingly, the data shows a lot of periodic behaviour, but no clear pattern. Even more interestingly, the data is "almost stable" (i.e. stabilises pointwise) when varying the field $L$ through an infinite tower of totally ramified extension. We didn't have enough time to explore this strange behaviour, but we will hopefully be able to do so in the future.

For me, this project was a great opportunity to learn a lot of interesting Mathematics (mostly algebraic in flavour) and work on an exciting and hard research problem. Not only did I learn a lot of technical results, but I got exposed to a whole world of modern algebra. I liked to work on an unsolved problem and, in particular, I loved the liberty to try any approach, even when most of them ultimately failed. This project truly confirmed my desire to do a PhD in Mathematics, probably something algebraic in nature. Again, I would like to thank Merton College for this wonderful opportunity and to express my deepest gratitude to everyone who made it possible.